Equivariant localizing invariants of simple varieties

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Abstract

We define a certain class of *simple varieties* over a field k by a constructive recipe and show how to control their (equivariant) truncating invariants. Consequently, we prove that on simple varieties:

- if $k = \overline{k}$ and $\operatorname{char} k = p$, the p-adic cyclotomic trace is an equivalence,
- if $k = \mathbb{Q}$, the Goodwillie–Jones trace is an isomorphism in degree zero,
- we can control homotopy invariant K-theory KH, which is equivariantly formal and determined by its topological counterparts.

Simple varieties are quite special, but encompass important singular examples appearing in geometric representation theory. We in particular show that both finite and affine Schubert varieties for GL_n lie in this class, so all the above results hold for them.

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0. Introduction

0.1. Motivation

The aim of this note is to record a technique for comparing and computing equivariant localizing invariants on a simple class of (often singular) varieties arising in geometric representation theory.

Let k be a base field and G be a nice algebraic group over it, for example an n-dimensional split torus $T = \mathbb{G}_m^n$ (more generally, any linearly reductive group G should qualify). We define classes $\mathcal{B}_k^G$ and $\mathcal{C}_k^G$ of certain proper G-equivariant k-schemes by a simple constructive recipe and call them *simple varieties*. Although such varieties are rather special, we nevertheless recover singular varieties of interest in geometric representation theory: *finite* and *affine Schubert varieties* for GL_n are the most prominent examples. In fact, these varieties are the original source of our motivation. The understanding of their (equivariant) cohomological invariants is an important topic in geometric representation theory; the most delicate information is controlled by their singularities. For instance, their intersection cohomology governs the representation theory of the algebraic group GL_n through the geometric Satake equivalence; this has further equivariant extensions. Such phenomena are expected to generalize to more complicated topological invariants.

One source of interesting invariants appearing in algebraic geometry are the *localizing invariants*. The most important examples come either from K-theory – such as algebraic K-theory $K(-)$ itself, its homotopy invariant version $KH(-)$, or its topological realization $K_{\mathrm{top}}(-)$ over $\mathbb{C}$; or from Hochschild homology – such as Hochschild homology $HH(-/k)$, negative cyclic homology $HC^-(-/k)$, topological cyclic homology $TC(-)$, and other variants. Localizing invariants can be evaluated on any derived stack through its category of perfect complexes. In particular, we can evaluate them on a global quotient X/G of any G-equivariant variety X, obtaining their *equivariant* versions.

0.2. Main results

The starting point of this project is to understand equivariant K-theory of affine Schubert varieties and trace maps from it, with a view towards geometric representation theory. In [Löw24] we described the relationship between T-equivariant K-theory and T-equivariant Hochschild homology of affine Schubert varieties after perfection in characteristic p. It turns out that our technique can be pushed further to control any truncating invariant (without perfection, in any characteristic).

This has two kinds of applications. On one hand, we can concretely control and compute such truncating invariants, most notably $KH(-)$. These are in particular equivariantly formal in a rather strong sense – while such formality statements were expected, they were missing until now. On the other hand, we can establish isomorphisms between different localizing invariants (without really computing them) as long as their difference is truncating – most notably between K-theory and variants of topological cyclic homology.

An abridged and simplified version of our present results goes as follows.

Theorem. *Let k be a field and G a nice (or linearly reductive*[1]*) group over it. We construct classes $\mathcal{B}_k^G \subseteq \mathcal{C}_k^G$ of G-equivariant projective k-schemes, which for example contain:*

- *projective spaces, Grassmannians, partial flag varieties,*
- *classical Schubert varieties in Grassmannians,*
- *affine Schubert varieties in the GL_n affine Grassmannian.*

For such varieties, the following statements hold:

- *If k is an algebraically closed field of characteristic p and $X \in \mathcal{C}_k^G$, then the mod-p cyclotomic trace induces an equivalence of ring spectra*

$$K^G(X; \mathbb{F}_p) \xrightarrow{\simeq} TC^G(X; \mathbb{F}_p).$$

Similarly for the p-adic version.

[1] See Remark 0.1 for discussion of the assumptions; the linearly reductive case is conditional.

- *If $k = \mathbb{Q}$ and $X \in \mathcal{B}^G_{\mathbb{Q}}$, then the Goodwillie–Jones trace induces a ring isomorphism*

$$K^G_0(X;\mathbb{Q}) \xrightarrow{\cong} (HC^-)^G_0(X/\mathbb{Q}).$$

  We further get a decomposition $K^G_i(X) \xrightarrow{\cong} KH^G_i(X) \oplus (HC^-)^G_i(X/\mathbb{Q})$ for all $i \geqslant 1$.

- *For any k and $X \in \mathcal{B}^G_k$, there is a natural ring isomorphism*

$$KH^G_0(X) \underset{K_0(\mathrm{pt})}{\otimes} K_\bullet(\mathrm{pt}) \xrightarrow{\cong} KH^G_\bullet(X)$$

  and $KH^G_0(X)$ is finite projective over $K^G_0(\mathrm{pt})$. In other words, the ring spectrum $KH^G(X)$ is connective and finite flat over $K^G(\mathrm{pt})$. When $k = \mathbb{C}$, we can further compare to topological K-theory: $KH^G_0(X) \cong K^G_{\mathrm{top},0}(X)$.

Proof. See Example 3.1, Lemma 3.10 and Lemma 3.14 for proofs that the mentioned varieties are simple. The postulated equivalences are then respectively given by Theorem 2.2, Theorem 2.11 (with Proposition 2.15) and Theorem 2.17 (with Proposition 2.23). □

Remark 0.1 (Assumptions on G)**.** At the current state of literature, the above theorem works for *nice* groups G over k, see §1.3. This covers algebraic tori $T = \mathbb{G}_m^{\times n}$, groups of roots of unity μ_ℓ and finite groups of order coprime to $\operatorname{char} k$ (as well as their extensions, subquotients and forms). The technical restriction boils down to [ES21, Corollaries 5.2.3 and 5.2.6], which is applicable to quotients by nice groups only. A cautious reader may want to restrict to nice G throughout the paper.

However, the upcoming work [LS25] extends the aforementioned results to quotients by *linearly reductive groups*. Conditionally on this extension, all our arguments go through without change for any linearly reductive group G over k. This does not make a difference if $\operatorname{char} k = p$ (where the notions of nice and linearly reductive coincide), but considerably extends the applicability if $\operatorname{char} k = 0$ to any reductive group (such as $G = GL_n$). We will precisely indicate in proofs the references to [LS25] needed for this generalization.

Remark 0.2 (Globality)**.** Let us highlight the following: our results are global. All the considered schemes are proper over k and our arguments stay within that realm. Even though localizing invariants satisfy Zariski (even Niesnevich) gluing, the desired isomorphisms do not hold Zariski locally. It is interesting that such isomorphisms nevertheless hold globally in naturally occurring singular examples.

0.3. Discussion and context

Let us emphasize that localizing invariants are highly sensitive to singularities – to the extent that their values are known only in the simplest singular cases. Even if the varieties in question have affine pavings (i.e. are cellular), there is no obvious method for the computations and comparisons of the localizing invariants we care about. The only way of attack is by detailed understanding of resolutions; this can be quite challenging. In effect, not many computations of equivariant K-theory nor KH in singular situations are known. We hope to contribute towards filling in this gap.

0.3.1. Discussion of results. Let us discuss the three comparisons from our main theorem in more detail. We emphasize that all these comparisons are compatible with the ring structures on the invariants at hand.

- The p-adic cyclotomic trace §2.1.
  The cyclotomic trace $K(-) \to TC(-)$ is the closest known approximation of algebraic K-theory by differential forms. In characteristic p with mod-p coefficients, this approximation is rather good. Nevertheless, it is known to be an equivalence only in specific local examples – see Remark 2.4 – and it is not an equivalence in general. Theorem 2.2 produces interesting (equivariant) singular projective examples where this is the case: for any $\mathcal{C}^G_k$, the map

$$K^G(X;\mathbb{F}_p) \xrightarrow{\simeq} TC^G(X;\mathbb{F}_p)$$

is still an equivalence – we are not aware of other nontrivial results of this form.

While the above generality is completely sufficient to cover the examples of our interest, the theorem easily generalizes to a slightly bigger class $\mathcal{C}_p$ of algebraic stacks, including some mixed-characteristic situations. We record this into a separate discussion in Theorem 2.8.

- Goodwillie–Jones trace §2.2.
When working over $\mathbb{Q}$, the cyclotomic trace specializes to the historically older Goodwillie–Jones trace $K(-) \to HC^-(-/\mathbb{Q})$ and we prove the isomorphisms

$$K_0^G(X;\mathbb{Q}) \xrightarrow{\cong} (HC^-)_0^G(X/\mathbb{Q})$$

for $X \in \mathcal{B}_k^G$ in Theorem 2.11. We further get a somewhat counterintuitive decomposition

$$K_i^G(X) \xrightarrow{\cong} KH_i^G(X) \oplus (HC^-)_i^G(X/\mathbb{Q}), \qquad i \geqslant 1.$$

in Proposition 2.15; this latter argument requires to use the supplementary truncating invariant $L_{\mathrm{cdh}}HC^-(-/\mathbb{Q})$ reviewed in Appendix A. We illustrate that this decomposition can be nontrivial in Example 3.7.

- Homotopy invariant K-theory §2.3.
The closest well-behaved approximation of algebraic K-theory of singular varieties is its homotopy invariant version KH. A reader interested in explicit ring presentations of equivariant invariants may want to focus on this part. Although no computations of KH in our setup appear in the literature, one expects to get reasonable results. This is indeed what we obtain: the natural identification

$$KH_0^G(X) \underset{K_0(\mathrm{pt})}{\otimes} K_\bullet(\mathrm{pt}) \xrightarrow{\cong} KH_\bullet^G(X)$$

from Theorem 2.17 shows that $KH^G(X)$ is equivariantly formal in a strong sense for any $X \in \mathcal{B}_k^G$, see also Remark 2.19. More conceptually, we prove that the ring spectrum $KH^G(X)$ is connective and finite flat over $K^G(\mathrm{pt})$. Equivariant formality statements of this form for KH are missing at the moment; we expect them to be useful for computations by Remark 2.22. We emphasize that $KH^G(X)$ is often explicitly computable – we do not seriously discuss this here, but see [Löw24, Remark 6.3, Examples 6.1 and 6.2] for some formulas.

The above formality has some immediate elementary consequences.
Firstly, for simple varieties from $\mathcal{B}_{\mathbb{C}}^G$, we deduce an isomorphism

$$KH_0^G(X) \cong K_{\mathrm{top},0}^G(X)$$

with topological K-theory of the analytification in Proposition 2.23. A posteriori, in these cases KH glues from affine pavings (since topological K-theory does); the whole $KH_\bullet^G(X)$ is determined by the independent geometric contribution of $K_{\mathrm{top},0}^G(X)$ and arithmetic contribution of $K_\bullet(\mathbb{C})$.
Secondly, simple varieties from $\mathcal{B}_{\mathbb{F}_q}^G$ satisfy Parshin's property by Example 2.29.

0.3.2. Strategy of proofs. In all three applications, our proof works as follows. We first express the desired statement in terms of some truncating invariant: in the first two cases, the fiber of the comparison map is truncating; in the third case KH is truncating to start with. We then check the statement on the point and consequently on the equivariant point BG; this can be usually done "by hand".

Once the above is settled, we are in a good shape. Indeed, we can control truncating invariants throughout the constructive steps defining simple varieties, so the results reduce to the case of BG. The subtle difference between the classes $\mathcal{B}_k^G$ and $\mathcal{C}_k^G$ morally depends on whether we wish to control the outputted homotopy groups one by one (as in the latter two applications), or if we are able to get a statement on the level of spectra (as in the first application).

To deduce our results for specific varieties (such as finite or affine Schubert varieties for GL_n), we just need to argue that they are are simple. These geometric arguments are completely independent from the homotopical machinery of localizing invariants.

We would like to point out that the presentation of the proof we are offering goes backwards – the ideas originate from abstracting the case of affine Schubert varieties.

0.3.3. Outlook. The above results should not be viewed as a definitive list of applications; it should rather give a taste of what is possible.

In one direction, one could try to come up with other naturally occurring varieties which happen to be simple. In particular, we do think that there are further example coming from geometric representation theory. Such endeavors can stay entirely withing the realm of classical algebraic geometry. One can also try to further optimize the definition of simple varieties.

In another direction, one could try to carry through more refined computations and comparisons of localizing invariants along the same recipe. For example, following the constructive definition of simple varieties, KH (as well as other truncation invariants) are actually computable. On a different note, all the comparison maps are compatible with motivic filtrations and our results should refine to this level – for instance, our results on KH should refine to the level of cdh-local motivic cohomology $\mathbb{Z}(j)^{\mathrm{cdh}}$ of [BEM25]. However, some technical care would be needed, as motivic cohomology complexes do not apriori exist as localizing invariants.

0.3.4. Other work. To close the introduction, we would like to give an idea of the complexity of K-theory of the singular varieties we are dealing with: see [CHWW13, Theorem 4.3] for computations of (non-equivariant) K-theory of the (affine) conic singularity $\{z^2 - xy = 0\} \subseteq \mathbb{A}^3_k$ over a field of characteristic zero; also see [PS21, §2.3, §2.4] for complementary results on finite quotient singularities. The projective conic coincidentally appears as the smallest singular affine Schubert variety for GL_2 – it lies in the class $\mathcal{B}_k$ where all our results apply.

0.4. Plan of the paper

This paper is structured as follows, separating the arguments into three logically independent parts. In §1 we define the classes of simple varieties $\mathcal{B}^G_k \subseteq \mathcal{C}^G_k$ by a constructive recipe, recall the necessary background on localizing invariants, and prove the main technical results – Theorems 1.17, 1.16 – which allow to control equivariant truncating invariants on simple varieties.
In §2 we discuss, one by one, the promised applications to various localizing invariants and equivalences between them – see in particular Theorems 2.2, 2.11, 2.17. These three applications can be read independently.
In §3, we supply explicit examples of simple varieties, starting from elementary ones and ending with finite and affine Schubert varieties for GL_n in Lemmas 3.10, 3.14. These final examples are truly interesting in geometric representation theory; all the results from the second section apply to them.

0.5. Acknowledgements

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1. Localizing and truncating invariants of simple varieties

We start by giving a constructive definition of the classes $\mathcal{B}^G_k \subseteq \mathcal{C}^G_k$ of simple varieties in §1.1, which are the main object of study in this paper. We continue by recalling the notion of localizing and truncating invariants in §1.2 as well as some group-theoretic preliminaries in §1.3. We then prove the technical result, Theorem 1.13 in §1.4, allowing us to control equivariant truncating invariants on simple varieties from $\mathcal{C}^G_k$. We close the section by useful variants valid on the smaller class $\mathcal{B}^G_k$, namely Theorems 1.16 and 1.17 in §1.5.

1.1. A class of simple varieties

Notation 1.1. Let k be a base field. We write $\mathrm{pt} = \operatorname{Spec} k$ and denote Sch_k the category of quasi-compact separated schemes over k. Let G be an affine group scheme over k. We then denote

Sch_k^G the category of G-equivariant quasi-compact separated schemes over k and equivariant maps between them. Any object X of Sch_k^G gives rise to the global quotient stack X/G. We denote $BG = \mathrm{pt}/G$ the classifying stack of G. Then $\mathrm{Vect}(X/G)$ and $\mathrm{Perf}(X/G)$ denote the categories of G-equivariant vector bundles and G-equivariant perfect complexes on X.

If R is an arbitrary base ring, we write $S = \operatorname{Spec} R$. If G is an affine group over R, we use analogous notation as above.

Notation 1.2. Given $X \in \mathrm{Sch}_k$, $\mathcal{F} \in \mathrm{QCoh}(X)$ and a dimension vector $d_\bullet = (d_1, \dots, d_m)$ of lenght m and total dimension $d := d_1 + \cdots + d_m$, we write $\mathrm{Flag}_X(\mathcal{F}, j_\bullet)$ for the relative flag bundle of type $d_\bullet$ parametrizing flags of vector bundle quotients of $\mathcal{F}$ with successive subquotients of ranks $d_1, \dots, d_m$. We reserve the notation $\mathrm{Flag}_X^*(\mathcal{F}, d_\bullet)$ for the dual convention parametrizing flags of subbundles. When $m = 1$, we get $\mathrm{Grass}_X(\mathcal{F}, d)$ resp. $\mathrm{Grass}_X^*(\mathcal{F}, d)$.

We give the constructive definition of simple varieties straightaway. We emphasize that this definition stays within the realm of classical algebraic geometry.

Definition 1.3 (Simple varieties $\mathcal{B}_k^G$)**.** Let $\mathcal{B}_k^G \subseteq \mathrm{Sch}_k^G$ be the full subcategory given by the following closure properties:

(1) (the point) The point pt with the trivial G action lies in $\mathcal{B}_k^G$. Moreover, $\mathcal{B}_k^G$ is closed on finite disjoint unions.

(2) (projective bundles) If $\mathcal{E} \in \mathrm{Vect}(X/G)$ is an equivariant vector bundle of rank $\geqslant 1$ everywhere[2], then
$$X \in \mathcal{B}_k^G \iff \mathbb{P}_X(\mathcal{E}) \in \mathcal{B}_k^G.$$
More generally, let $d_\bullet$ be any dimension vector of total dimension d and assume that $\mathcal{E}$ has rank $\geqslant d$ everywhere. Then
$$X \in \mathcal{B}_k^G \iff \mathrm{Flag}_X(\mathcal{E}, d_\bullet) \in \mathcal{B}_k^G$$

(3) (stratified projective bundles with nonempty fibers) Let $X \in \mathrm{Sch}_k$ and $\mathcal{F} \in \mathrm{QCoh}(X/G)$ be an equivariant quasi-coherent sheaf on X that can be written as a cokernel of a map of equivariant vector bundles. Assume that $\mathcal{F}$ has rank $\geqslant 1$ everywhere. Then
$$\mathbb{P}_X(\mathcal{F}) \in \mathcal{B}_k^G \implies X \in \mathcal{B}_k^G.$$
More generally, let $d_\bullet$ be any dimension vector of total dimension d and assume that $\mathcal{F}$ has rank $\geqslant d$ everywhere. Then
$$\mathrm{Flag}_X(\mathcal{F}, d_\bullet) \in \mathcal{B}_k^G \implies X \in \mathcal{B}_k^G.$$

(4) (3-out-of-4 for split abstract blowups) Take a noetherian abstract blowup square (X, Y, Z, E) in Sch_k^G, meaning an equivariant pullback square of noetherian schemes over k
$$\begin{tikzcd} Y \arrow[d] & E \arrow[l, hook'] \arrow[d] \\ X & Z, \arrow[l, hook'] \end{tikzcd}$$
where $X \hookleftarrow Z$ is a closed immersion while $Y \to X$ is proper and isomorphism over $X \backslash Z$. Assume that the square is *split* in the sense that either $E \to Y$ admits a retraction or $Y \to X$ admits a section. If three of its terms lie in $\mathcal{B}_k^G$, then the fourth term lies in $\mathcal{B}_k^G$ as well.

Definition 1.4 (Simple varieties $\mathcal{C}_k^G$)**.** We define a bigger class $\mathcal{C}_k^G \supseteq \mathcal{B}_k^G$ by the closure properties (1), (2), (3) and:

(4') (3-out-of-4 for abstract blowups) Take a noetherian abstract blowup square (X, Y, Z, E) in Sch_k^G. If three of its terms lie in $\mathcal{C}_k^G$, then the fourth term lies in $\mathcal{C}_k^G$ as well.

[2] The rank conditions here and below are equivalent to the associated projectivization having nonempty fibers.

Notation 1.5. In particular, for trivial G we get the subcategories $\mathcal{B}_k \subseteq \mathcal{C}_k$ of Sch_k. Already this case is interesting.

Remark 1.6. If $G' \to G$ is a group homomorphism and $X \in \mathcal{B}_k^G$, we get $X \in \mathcal{B}_k^{G'}$ by restricting the action. In particular, $X \in \mathcal{B}_k$. Similarly for the bigger class $\mathcal{C}_k^G$.

Remark 1.7. Since Definitions 1.3, 1.4 are sufficient for our applications, we in general stick to them to minimize the technical load.

Nevertheless, they can be further optimized. Depending on the concrete application, we can even add more base cases to (1). For example, see Definition 2.6 in §2.1.2 for an absolute class $\mathcal{C}_p$ depending only on a prime p relevant for the cyclotomic trace, or Remark 2.20 for a discussion of KH of simple varieties over an arbitrary base ring R.

1.2. Localizing and truncating invariants

We now recall the invariants we wish to control. Let E be any k-linear *localizing invariant* valued in spectra in the sense of [LT19, Definition 2 and Remark 1.18]. By definition, $E(-)$ is a functor of ∞-categories

$$E(-) : \mathrm{Cat}_k^{\mathrm{perf}} \to \mathrm{Sp}$$

which sends exact sequences to fiber sequences. In particular, if $\mathcal{D} \in \mathrm{Cat}_k^{\mathrm{perf}}$ has a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_j \rangle_{j=0}^m$, we get a natural decomposition $E(\mathcal{D}) \simeq \bigoplus_{j=0}^m E(\mathcal{D}_j)$.

A localizing invariant is called *finitary* if it further commutes with filtered colimits. Many important localizing invariants have this property (and sometimes it is included into the definition). Also see [Tam18; LT19; HSS17; BGT13].

A localizing invariant E is called *truncating* if it satisfies [LT19, Definition 3.1]: for every connective $\mathbb{E}_1$-ring spectrum A, the canonical map $E(A) \to E(\pi_0(A))$ is an equivalence. This further forces nilinvariance and cdh descent in various degrees of generality [LT19; KR24; ES21].

By convention, we evaluate E on any derived stack X via its category of perfect complexes

$$E(X) := E(\mathrm{Perf}(X)).$$

In particular, given any G-equivariant scheme $X \in \mathrm{Sch}_k^G$, we denote

$$E^G(X) := E(X/G) := E(\mathrm{Perf}(X/G))$$

and freely switch between these notations.

For any derived stack X, the category $\mathrm{Perf}(X)$ carries a symmetric monoidal structure $\otimes$. All the localizing invariants we consider are lax monoidal: they in particular turn this into an $\mathbb{E}_\infty$-ring structure on the outputted spectrum $E(X)$, whose homotopy groups thus form a graded-commutative ring $E_\bullet(X)$. For brevity, we call such localizing invariants *multiplicative*.

Recollection 1.8. The following are k-linear localizing invariants:

- non-connective algebraic K-theory $K(-)$, it is finitary,
- topological Hochschild homology $THH(-)$, it is finitary,
- Hochschild homology of k-linear categories $HH(-/k)$, it is finitary,
- negative cyclic homology of k-linear categories $HC^-(-/k)$,
- topological cyclic homology $TC(-)$.

Furthermore, the following are k-linear truncating invariants:

- homotopy K-theory $KH(-)$ of k-linear categories, it is finitary,
- the fiber of the cyclotomic trace $K^{\mathrm{inf}}(-)$,
- periodic cyclic homology of k-linear categories $HP(-/k)$,
- topological K-theory of $\mathbb{C}$-linear categories $K_{\mathrm{top}}(-)$, it is finitary.

Proof. For $K(-)$, $THH(-)$, see [BGT13], [Tam18, Example 17]. For $HH(-/k)$, see [Hoy18], [HSS17]; it commutes with filtered colimits by definition. Further see [LT19, p. 30] for a discussion of $HH(-/k)$, $HC^-(-/k)$, $HP(-/k)$ as localizing invariants and [LT19, proof of Corollary 3.6] for $TC(-)$ and the truncating invariant $K^{\mathrm{inf}}(-)$. For $K_{\mathrm{top}}(-)$ as a finitary localizing invariant see [Bla16, Theorem 1.1]; it is truncating by [Kon21].

Also $KH(-)$ is a truncating invariant by [LT19, Proposition 3.14]; it commutes with filtered colimits as $K(-)$ does by swapping colimits in its definition. □

Notation 1.9. We use the following notation:

- $K(-;\mathbb{F}_p) := K(-)/p$ for the mod-p algebraic K-theory,
- $K(-;\mathbb{Z}/p^j)$ and $K(-;\mathbb{Z}_p)$ for its mod-p^j and p-adic versions,
- $K(-;\mathbb{Q}) := K(-)_{\mathbb{Q}}$ for rationalized K-theory.

See [TT90, §9.3] for details. Given any localizing invariant E in place of K-theory, we use the analogous notations $E(-;\mathbb{F}_p)$, $E(-;\mathbb{Z}/p^j)$, $E(-;\mathbb{Z}_p)$ and $E(-;\mathbb{Q})$.

The above discussion also works over an arbitrary base ring R in place of k; we will use this in a few side remarks.

1.3. An interlude on groups and representations

We now discuss the groups that we eventually want to consider. The results of this paper work best for nice groups over fields. However, conditionally on the upcoming work [LS25], our results generalize to linearly reductive groups over fields (giving broader applicability in characteristic zero). In a different direction, restricting to diagonalizable groups only, our results work over more general base rings.

1.3.1. Nice and diagonalizable groups. Let M be an abstract abelian group. Then its group algebra $R[M]$ is naturally a commutative and cocommutative Hopf algebra, and we write $D(M) = \mathrm{Spec}(R[M])$ for the corresponding affine group scheme. An affine group scheme G over R is called *diagonalizable* if it is of the form $D(M)$ for some abelian group M.

The diagonalizable group $D(M)$ is of finite type if and only if M is finitely generated. By the classification of finitely generated abelian groups, diagonalizable groups of finite type are given by finite products of the multiplicative group $\mathbb{G}_m$ and the groups of finite order roots of unity μ_{p^a} of a prime power p^a.

An affine group scheme G over R is of *multiplicative type* if it is diagonalizable fpqc locally on R.

An affine group scheme G of finite type over R is called *nice*, if it is an extension of a closed and open normal subgroup G^0 of multiplicative type by an étale group scheme of order invertible in R. It is called *linearly reductive* if the functor $(-)^G$ of invariants is exact. Nice groups are always linearly reductive.

1.3.2. Linearly reductive groups over fields. Let G be a linearly reductive affine group scheme of finite type over a field k. In concrete terms [AHR19, Remark 2.3], such G is given as follows:

(i) if $\mathrm{char}\, k = 0$, this is equivalent to saying that G is an extension of a connected reductive group by a finite group,

(ii) if $\mathrm{char}\, k = p$ is positive, this is equivalent to saying that G is nice, i.e. an extension of a group of multiplicative type by an étale group whose order is not divisible by p.

In characteristic p, the notions of nice and linearly reductive groups coincide. In characteristic zero, reductive groups form a bigger class (e.g. GL_n is linearly reductive but not nice).

1.3.3. Representations and decomposability. In order to compute equivariant localizing invariants of the point, we need to control the category $\mathrm{Perf}(BG)$. We denote by $\mathrm{Rep}_k(G)$ the category of finite dimensional algebraic representations of G; it is tautologically equivalent to the category $\mathrm{Vect}(BG)$ of vector bundles on its classifying stack. We write $R(G) = K_0(BG)$ for the representation ring of G.

The above categories become very easy in the following naturally occurring situation.

Discussion 1.10. We say that an affine group G scheme over a ring R has *decomposable representation theory* if there is an index set I and an equivalence in $\mathrm{Cat}_R^{\mathrm{perf}}$

$$\mathrm{Perf}(R/G) \simeq \bigoplus_I \mathrm{Perf}(R). \tag{1.1}$$

The following groups give the most notable examples.

(i) Let G be a linearly reductive group over a field k. Then the category $\mathrm{Rep}_k(G)$ is semisimple; it is an infinite direct sum of copies of $\mathrm{Vect}(\mathrm{pt})$ labeled by the irreducibles in $\mathrm{Rep}_k(G)$. Consequently, G has decomposable representation theory with $\mathrm{Perf}(BG) \simeq \bigoplus_{\mathrm{Irr}(G)} \mathrm{Perf}(k)$.

(ii) Let $G = D(M)$ is a diagonalizable group over $S = \operatorname{Spec} R$. Representations of G correspond to M-graded quasi-coherent sheaves on S. The latter abelian category decomposes as a direct product of copies of quasi-coherent sheaves on R over the index set M. Consequently, G has decomposable representation theory with $\mathrm{Perf}(S/G) \simeq \bigoplus_M \mathrm{Perf}(S)$.

In these examples, the natural monoidal structure on the left-hand side corresponds to the convolution on the right-hand side.

Decomposition (1.1) has the following simple consequence: the value of finitary localizing invariants on the equivariant base is determined by its value on R.

Lemma 1.11. *Let G be a group over R with decomposable representation theory and E a finitary localizing invariant. Then*

$$E^G(R) \simeq \bigoplus_I E(R).$$

Proof. Since $E(-)$ commutes with finite direct sums and filtered colimits, it is compatible with infinite direct sums, so we conclude from (1.1). ☐

Remark 1.12. In applications, one can often check the outcome of Lemma 1.11 even in cases when E is not finitary per se. More precisely, it is often sufficient to know that $E(\mathrm{pt})$ is sufficiently bounded. See Lemmata 2.5 and 2.12.

1.4. Truncating invariants of simple varieties

Let k be a field and G a nice group over it. Conditionally, the results generalize to any linearly reductive group G over k. The following theorem then gives control over truncating invariants on $\mathcal{C}_k^G$. It is one of the main technical observations of this paper. Also see §1.5 for further variations.

Theorem 1.13. *Let E be a k-linear truncating invariant with $E^G(\mathrm{pt}) \simeq 0$. Then for all $X \in \mathcal{C}_k^G$ we have*

$$E^G(X) \simeq 0.$$

Proof. We check that $E^G(X) \simeq 0$ is stable under the closure properties from Definition 1.4.

(1) By assumption, $E^G(\mathrm{pt}) \simeq 0$; the compatibility with finite disjoint unions is clear as E commutes with finite direct sums.

(2) For $\mathbb{P}_X(\mathcal{E})$, this follows from the projective bundle formula for localizing invariants. More precisely, $\mathrm{Perf}(\mathbb{P}_{X/G}(\mathcal{E}))$ has a semi-orthogonal decomposition into n copies of $\mathrm{Perf}(X/G)$ by the classical results of [Tho88]; see also [Kha20, Theorem 3.3 and Corollary 3.6]. Since localizing invariants send semi-orthogonal decompositions to direct sums, we have $E^G(\mathbb{P}_X(\mathcal{E})) \simeq \bigoplus_{i=0}^{n-1} E^G(X)$, giving the desired equivalence.

For the case of a partial flag variety $\mathrm{Flag}_X(\mathcal{E}, d_\bullet)$, consider the corresponding full flag variety $\mathrm{Flag}_X(\mathcal{E}, d'_\bullet)$ where $d'_\bullet = (1, \dots, 1)$ of total dimension d. Then we have natural maps

$$\begin{tikzcd} \mathrm{Flag}_X(\mathcal{E}, d'_\bullet) \arrow[r, "g"'] \arrow[rr, bend left, "h"] & \mathrm{Flag}_X(\mathcal{E}, d_\bullet) \arrow[r, "f"'] & X \end{tikzcd}$$

and both g and h factor as towers of equivariant projective bundles, reducing to the previous case.

(3) By the assumption on $\mathcal{F}$, we can find a two term complex of equivariant vector bundles $\mathcal{E}_\bullet = [\mathcal{E}_1 \to \mathcal{E}_0]$ with $\mathrm{coker}(\mathcal{E}_1 \to \mathcal{E}_0) \cong \mathcal{F}$. We can then take its derived projectivization $Y = \mathbb{P}_X(\mathcal{E}_\bullet)$ in the sense of [Jia22b; Jia22a]. This is a derived scheme with classical truncation $Y_{\mathrm{cl}} = \mathbb{P}_X(\mathcal{F})$. Since $\mathcal{E}_\bullet$ has Tor-amplitude $[1, 0]$, we obtain the semi-orthogonal decomposition of [Jia23, Theorem 3.2 and Remark 1.1], [Jia22b, Theorem 7.5]. In particular, as the rank of $\mathcal{F}$ is $\geqslant 1$ everywhere, the pullback realizes $\mathrm{Perf}(X/G)$ as a semiorthogonal summand in $\mathrm{Perf}(Y/G)$ – note that this is independent of the characteristic of k (in fact works over any base) by [Jia23, Remark 1.1], [Jia22b, Theorem 7.5] since we are treating only derived projectivizations at the moment. Since localizing invariants send semi-orthogonal decompositions to direct sums, $E^G(Y) \simeq 0 \implies E^G(X) \simeq 0$. Moreover, as $E(-)$ is truncating, we have $E^G(Y_{\mathrm{cl}}) \simeq E^G(Y)$ by [ES21, Corollary 5.2.3] – if G is merely linearly reductive, this needs the forthcoming extension [LS25]. Altogether, this proves the first statement.

For the second statement, consider again the diagram

$$\begin{tikzcd} \mathrm{Flag}_X(\mathcal{F}, d'_\bullet) \arrow[r, "g"'] \arrow[rr, bend left, "h"] & \mathrm{Flag}_X(\mathcal{F}, d_\bullet) \arrow[r, "f"'] & X \end{tikzcd}$$

where $\mathrm{Flag}_X(\mathcal{F}, d'_\bullet)$ is the stratified full flag variety bundle for $d' = (1, \dots, 1)$ of total dimension d. Now g is a tower of honest projective bundles, so we know $E^G(\mathrm{Flag}_X(\mathcal{F}, d_\bullet)) \simeq 0$ if and only if $E^G(\mathrm{Flag}_X(\mathcal{F}, d'_\bullet)) \simeq 0$. Moreover, h is a tower of stratified projective bundles, so we already know that $E^G(\mathrm{Flag}_X(\mathcal{F}, d'_\bullet)) \simeq 0 \implies E^G(X) \simeq 0$ by iterative use of the previous step. Hence we conclude.

(4) Any truncating invariant sends G-equivariant abstract blowup squares to homotopy fiber squares by [ES21, Corollary 5.2.6] – if G is merely linearly reductive, this needs the fothcoming extension [LS25]. The statement follows. ☐

Remark 1.14. The discussion of (1) and (2) clearly works for any localizing invariant. However, for both (3) and (4'), it is crucial that E is truncating. The closure properties (3) and (4') are important, as they allow us to go down and reach singular varieties.

Corollary 1.15. *Let $E \to F$ be a map of k-linear finitary truncating invariants. If it induces an equivalence $E(\mathrm{pt}) \simeq F(\mathrm{pt})$, then it induces an equivalence $E^G(X) \simeq F^G(X)$ for any $X \in \mathcal{C}_k^G$.*

Proof. The fiber $\mathrm{fib}(E \to F)$ is a k-linear finitary truncating invariant which vanishes on pt. We conclude by Lemma 1.11 and Theorem 1.13. ☐

The assumption that E, F are finitary is used solely to commute E through the decomposition (1.10) via Lemma 1.11. As we already mentioned in Remark 1.12, this can often be done by hand in more general situations.

1.5. Degreewise versions

If we restrict attention only to the smaller class $\mathcal{B}_k^G$, Theorem 1.13 works degreewise. This is a very useful variant – there are interesting maps of localizing invariants which induce isomorphisms only in certain degrees.

As before, assume G is a nice group over a field k. Conditionally, the results generalize to any linearly reductive group G over k.

Theorem 1.16. *Fix $i \in \mathbb{Z}$ and let E be a k-linear truncating invariant with $E_i^G(\mathrm{pt}) \simeq 0$. Then for all $X \in \mathcal{B}_k^G$ we have*

$$E_i^G(X) \simeq 0.$$

More generally, fix $i \in \mathbb{Z}$ and let E and F be k-linear truncating invariants together with a map $E \to F$ which induces an isomorphism $E_i^G(\mathrm{pt}) \simeq F_i^G(\mathrm{pt})$. Then for all $X \in \mathcal{B}_k^G$ we have

$$E_i^G(X) \simeq F_i^G(X).$$

Proof. We focus on the latter statement, the first being a special case for F trivial. It is enough to check that $E_i^G(X) \simeq F_i^G(X)$ is stable under the closure properties from Definition 1.3. This follows by the arguments from the proof of Theorem 1.13. Indeed, the desired isomorphism is stable under taking direct sums and direct summands in $\mathrm{Cat}_k^{\mathrm{perf}}$, so it is stable under (1) and (2). Since $E_i(-)$ and $F_i(-)$ disregard derived structures on stacks in the necessary generality [ES21, Corollary 5.2.3], resp. [LS25] in the linearly reductive case, the compatibility with direct summands shows the stability under (3). Since $E_i(-)$ and $F_i(-)$ turn split abstract blowup squares of stacks into split homotopy fiber squares in the necessary generality [ES21, Corollary 5.2.6], resp. [LS25], we deduce the stability on (4). $\square$

In a similar vein, the value of any multiplicative truncating invariant E on some $X \in \mathcal{B}_k^G$ is determined by the degree zero part $E_0^G(X)$, which is well-controlled.

Theorem 1.17. *Let E be a k-linear finitary truncating invariant, which is multiplicative. Then for all $X \in \mathcal{B}_k^G$ the natural map induces a graded ring isomorphism*

$$E_0^G(X) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt}) \xrightarrow{\cong} E_\bullet^G(X). \tag{1.2}$$

Furthermore, the commutative ring $E_0^G(X)$ is a finite projective module over $E_0^G(\mathrm{pt})$.

Proof. Since E is multiplicative, the universal property of tensor products induces is a natural ring homomorphism

$$E_0(-) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt}) \to E_\bullet(-) \tag{1.3}$$

and we only need to check that it is an isomorphism on the underlying abelian groups. We check that this statement is stable under the closure properties from Definition 1.3.

For non-equivariant pt, the map (1.3) is tautologically an isomorphism. Since E is finitary, both sides of (1.3) commute with the infinite direct sum (1.1), giving

$$E_0^G(\mathrm{pt}) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt}) \xrightarrow{\cong} E_\bullet^G(\mathrm{pt}). \tag{1.4}$$

For the same reason, the statement is stable under disjoint unions. Altogether, we have checked the closure property (1).

Given $\mathcal{D} \in \mathrm{Cat}_k^{\mathrm{perf}}$ with a semi-orthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_j \rangle_{j=0}^m$ and $i \in \mathbb{Z}$, we have

$$E_i(\mathcal{D}) = \bigoplus_{j=0}^m E_i(\mathcal{D}_j), \qquad \text{hence} \qquad E_0(\mathcal{D}) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt}) = \bigoplus_{j=0}^m E_0(\mathcal{D}_j) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt}),$$

so both sides of (1.3) send semi-orthogonal decompositions to direct sums.

In particular, stability under (2) follows from the semi-orthogonal decompositions for projective bundles. Since E is truncating, the closure property (3) follows from the the semi-orthogonal decompositions of derived projectivizations [Jia23] together with the independence of $E_i(-)$ on derived structures [ES21, Corollary 5.2.3], [LS25].

Finally, since E is truncating, any noetherian split abstract blowup (X, Y, Z, E) induces a long exact sequence on homotopy groups of E by [ES21, Corollary 5.2.3], [LS25], which falls apart into split short exact sequences

$$0 \to E_i^G(X) \to E_i^G(Y) \oplus E_i^G(Z) \to E_i^G(E) \to 0, \qquad i \in \mathbb{Z}\,.$$

In particular, we get the split short exact sequence

$$0 \to E_0^G(X) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt}) \to (E_0^G(Y) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt})) \oplus (E_0^G(Z) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt})) \to E_0^G(E) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt}) \to 0.$$

Altogether, both sides of (1.3) behave compatibly under the closure property (4).

Finally, in each step (1)-(4) we are only passing to finite direct sums or retracts. It is thus clear that the property of $E_0^G(X)$ being finite projective over $E_0^G(\mathrm{pt})$ is preserved. $\square$

Remark 1.18. The statements of Theorems 1.16, 1.17 are not true for the bigger class $\mathcal{C}_k^G$ by Example 3.4.

Remark 1.19. The isomorphism from Theorem 1.17 refines to

$$E_0^G(X) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt}) \xrightarrow{\cong} E_0^G(X) \underset{E_0^G(\mathrm{pt})}{\otimes} E_\bullet^G(\mathrm{pt}) \xrightarrow{\cong} E_\bullet^G(X).$$

Indeed, we have natural maps as indicated; the first one is also an isomorphism by (1.4).

For the next remark, recall that a map of $\mathbb{E}_\infty$-ring spectra $A \to B$ is called *(finite) faithfully flat* if the map of commutative rings $\pi_0(A) \to \pi_0(B)$ is (finite) faithfully flat and $\pi_\bullet(B) \cong \pi_0(B) \otimes_{\pi_0(A)} \pi_\bullet(A)$. In this language, we can rephrase the above results as follows.

Remark 1.20. Concisely, $E^G(X)$ is faithfully flat over $E(\mathrm{pt})$. It is finite faithfully flat over $E^G(\mathrm{pt})$.

2. Applications to concrete localizing invariants

We now present several instances of the above theorems applied to concrete localizing invariants. The most interesting applications concern global situations in positive and mixed characteristic where the p-adic cyclotomic trace becomes an equivalence §2.1; situations over the rationals where the Goodwillie–Jones trace induces an isomorphism in degree zero §2.2; and situations where homotopy-invariant K-theory becomes equivariantly formal §2.3. The lastly mentioned formality gives, for instance, a way to compute KH of complex simple varieties from their topological K-theory K_{top}.

2.1. The p-adic cyclotomic trace

As a first application, we show in Theorem 2.2 that the mod-p cyclotomic trace is an equivalence on simple varieties with nice group actions; this already covers the geometric examples of our interest.

We then explain a mild generalization of the result including certain henselian situations in Theorem 2.8 – a reader interested in varieties over a field can skip this variant.

Setup 2.1. Throughout this section, assume k is an algebraically closed field of characteristic p. Let G be a nice group scheme over k.

2.1.1. The p-adic cyclotomic trace on simple varieties. We give the following application of Theorem 1.13 to simple varieties from $\mathcal{C}_k^G$.

Theorem 2.2 (Mod-p cyclotomic trace)**.** *For any $X \in \mathcal{C}_k^G$, the mod-p cyclotomic trace induces an equivalence*

$$K^G(X; \mathbb{F}_p) \xrightarrow{\simeq} TC^G(X; \mathbb{F}_p).$$

Proof. Note that the fiber $K^{\mathrm{inf}}(-; \mathbb{F}_p)$ is a truncating invariant. We check that $K^{\mathrm{inf}}(BG; \mathbb{F}_p) \simeq 0$ in Lemma 2.5 below, hence $K^{\mathrm{inf}}(X/G; \mathbb{F}_p) \simeq 0$ for any $X \in \mathcal{C}_k^G$ by Theorem 1.13 as desired. ☐

Remark 2.3 (p-adic cyclotomic trace)**.** The same conclusion formally follows for mod-p^j and p-adic versions: for any $X \in \mathcal{C}_k^G$, we have

$$\forall j \in \mathbb{N},\ K^G(X; \mathbb{Z}/p^j) \xrightarrow{\simeq} TC^G(X; \mathbb{Z}/p^j) \qquad \text{and} \qquad K^G(X; \mathbb{Z}_p) \xrightarrow{\simeq} TC^G(X; \mathbb{Z}_p).$$

Remark 2.4 (Non-equivariant case is interesting)**.** The above statement is interesting already non-equivariantly: for any $X \in \mathcal{C}_k$, it gives an equivalence $K(X; \mathbb{F}_p) \xrightarrow{\simeq} TC(X; \mathbb{F}_p)$. In general, it is known that the cyclotomic trace $K(X; \mathbb{F}_p) \to TC(X; \mathbb{F}_p)$ realizes the target as the étale sheafification of the source by [CMM21, Theorem 6.3]; also see [GH99] for results in the smooth case. In other words, our Theorem 2.2 asserts that the étale sheafification is not necessary after evaluation on $X \in \mathcal{C}_k$. To us, this is far from obvious: we do not know how to deduce our result by studying the étale site of X.

To complete the proof of the theorem, we still need to supply the case of the equivariant point.

Lemma 2.5. *We have an equivalence*

$$K^G(\mathrm{pt};\mathbb{F}_p) \xrightarrow{\simeq} TC^G(\mathrm{pt};\mathbb{F}_p).$$

In other words, $K^{\mathrm{inf}}(BG;\mathbb{F}_p) \simeq 0$.

Proof. Since k is an algebraically closed field of characteristic p, the cyclotomic trace

$$K(\mathrm{pt};\mathbb{F}_p) \xrightarrow{\simeq} TC(\mathrm{pt};\mathbb{F}_p) \tag{2.1}$$

is an equivalence. Indeed, $K(\mathrm{pt})$ lives in degrees $\geqslant 0$ and all the higher K-groups are $\mathbb{Z}[\frac{1}{p}]$-modules [Kra80, Corollary 5.5. and Example (1) below it]. Since $K_0(\mathrm{pt}) \cong \mathbb{Z}$, we see that $K(\mathrm{pt};\mathbb{F}_p)$ is concentrated in degree 0 with value $\mathbb{F}_p$. At the same time, since k is algebraically closed, $TC(\mathrm{pt};\mathbb{F}_p)$ also lives in degree 0 with value $\mathbb{F}_p$ by [KN18, Example 7.4 and Remark 7.5]. The trace map clearly induces an isomorphism between them.

To treat the equivariant case, consider the decomposition (1.1). Since $THH(-)$ is finitary, we get an equivalence

$$THH(BG) \simeq \bigoplus_{\mathrm{Irr}(G)} THH(\mathrm{pt}). \tag{2.2}$$

Note that $THH(k)$ is connective (this being the case for THH of any connective ring spectrum in place of k), so both sides of (2.2) are connective. By general theory, $THH(-)$ carries a natural structure of a cyclotomic spectrum; the forgetful map from cyclotomic spectra to spectra preserves colimits [CMM21, §2.1, p. 416]. Altogether, (2.2) holds in the category CycSp of cyclotomic spectra, and hence in the category $\mathrm{CycSp}_{\geqslant 0}$ of connective cyclotomic spectra. Now, by [CMM21, Theorem 2.7], the functor $TC(-;\mathbb{F}_p) : \mathrm{CycSp}_{\geqslant 0} \to \mathrm{Sp}$ commutes with colimits. Altogether,

$$TC(BG;\mathbb{F}_p) \simeq \bigoplus_{\mathrm{Irr}(G)} TC(\mathrm{pt};\mathbb{F}_p).$$

Since K-theory is finitary, the decomposition (1.1) also gives

$$K(BG;\mathbb{F}_p) \simeq \bigoplus_{\mathrm{Irr}(G)} K(\mathrm{pt};\mathbb{F}_p).$$

All the above identifications are compatible with the trace map, so we reduce to the case of pt treated in (2.1) above. □

2.1.2. Strictly henselian base rings. In fact, Theorem 2.2 can be easily generalized to include certain strictly henselian situations in both equal and mixed characteristic as base cases. To this end, we introduce the following absolute class of simple stacks $\mathcal{C}_p$ depending only on a prime number p, which is given by allowing more general base examples in Definition 1.3. We use this as an opportunity to rephrase the rest of the construction in a stacky way.

We denote $\mathrm{dStk}^{\mathrm{ans}}$ the category of derived algebraic stacks with nice stabilizers, see [BKRS22, §A.1], [KR24, §2.4 and §2.5]. We also recall that the notions of abstract blowup squares [BKRS22, Definition 2.1.3], Nisnevich squares [BKRS22, Definition 2.1.1], derived projectivizations [Jia22b; Jia22a; Jia23] work in this generality. We then define the class $\mathcal{C}_p \subseteq \mathrm{dStk}^{\mathrm{ans}}$ as follows.

Definition 2.6. Let p be a fixed prime. Let $\mathcal{C}_p \subseteq \mathrm{dStk}$ be the class of derived algebraic stacks given by the following closure properties:

(0) (independence on nilpotence) Let $\mathcal{X} \in \mathrm{dStk}^{\mathrm{ans}}$. Then $\mathcal{X} \in \mathcal{C}_p \iff \mathcal{X}_{\mathrm{cl}} \in \mathcal{C}_p \iff \mathcal{X}_{\mathrm{red}} \in \mathcal{C}_p$.

(1) (base cases) Let $S = \operatorname{Spec} R$ be the spectrum of a strictly henselian, weakly regular, stably coherent ring R of residue characteristic p. Let G be a group scheme over S with decomposable representation theory. Then $S/G \in \mathcal{C}_p$. Moreover, $\mathcal{C}_p$ is closed on finite disjoint unions.

(2) (projective bundles) Let $\mathcal{X} \in \mathrm{dStk}$ and $\mathcal{E} \in \mathrm{Vect}(\mathcal{X})$ of rank $\geqslant 1$ everywhere, then $\mathbb{P}_{\mathcal{X}}(\mathcal{E}) \in \mathcal{C}_p \iff \mathcal{X} \in \mathcal{C}_p$. More generally, let $d_\bullet$ be a dimension vector of total dimension d and assume that $\mathcal{E}$ has rank $\geqslant d$ everywhere, then $\mathcal{X} \in \mathcal{C}_p \iff \mathrm{Flag}_{\mathcal{X}}(\mathcal{E}, d_\bullet) \in \mathcal{C}_p$.

(3) (stratified projective bundles) Let $\mathcal{X} \in \mathrm{dStk}$ and $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})$ be of the form $\mathcal{F} = H_0(\mathcal{E})$ for some $\mathcal{E} \in \mathrm{Perf}^{\geqslant 0}(\mathcal{X})$. Assume $\mathcal{F}$ has rank $\geqslant 1$ everywhere, then $\mathbb{P}_{\mathcal{X}}(\mathcal{F}) \in \mathcal{C}_p \implies \mathcal{X} \in \mathcal{C}_p$. More generally, let $d_\bullet$ be a dimension vector of total dimension d and assume that $\mathcal{F}$ has rank $\geqslant d$ everywhere, then $\mathrm{Flag}_{\mathcal{X}}(\mathcal{F}, d_\bullet) \in \mathcal{C}_p \implies \mathcal{X} \in \mathcal{C}_p$.

(4) (cdh descent) Given a noetherian abstract blowup square $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{E})$ in dStk such that three of its terms lie in $\mathcal{C}_p$, the fourth one does as well. Similarly for arbitrary Nisnevich squares.

Example 2.7. Let us mention some examples of stacks in $\mathcal{C}_p$.

(i) If G is a nice group over any algebraically closed field k of characteristic p and $X \in \mathcal{C}_k^G$, then $\mathcal{X} := X/G \in \mathcal{C}_p$.

(ii) We allow more general affine base rings R to lie in $\mathcal{C}_p$ in the closure property (1). In particular, R can be any regular noetherian strictly henselian ring such as the power series ring $k[\![t]\!]$ or the ring of integers $\breve{\mathbb{Z}}_p$ of the completion of the maximal unramified extension of $\mathbb{Q}_p$. More generally, R can be any strictly henselian valuation ring by [AMM22, Corollary 2.3].

(iii) The remaining closure properties then allow to combine the above: for example, the blowup of such regular noetherian strictly henselian R in its special point again lies in $\mathcal{C}_p$.

We then have the following generalization of Theorem 2.2 and Lemma 2.5 to the class $\mathcal{C}_p$.

Theorem 2.8. *For any $\mathcal{X} \in \mathcal{C}_p$, the cyclotomic trace induces an equivalence*

$$K(\mathcal{X}; \mathbb{F}_p) \xrightarrow{\simeq} TC(\mathcal{X}; \mathbb{F}_p).$$

Similarly for the mod-p^j and p-adic variants.

Proof. The independence on nilpotent structures (0) is [ES21]. The base cases (1) follow from Lemma 2.9 below. Then (2) and (3) are same as in Theorem 1.17 using that the semiorthogonal decompositions for projective bundles and derived projectivizations work over $\mathbb{Z}$, see [Jia22b, Theorem 7.5] and [Jia23, Remark 1.1]. Finally, (4) follows from [ES21, Corollary 5.2.6]. $\square$

Lemma 2.9. *Let $S = \operatorname{Spec} R$ be the spectrum of a strictly henselian, weakly regular, stably coherent ring R of residue characteristic p. Consider a group G over S with decomposable representation theory. Then the cyclotomic trace induces an equivalence*

$$K(S/G; \mathbb{F}_p) \xrightarrow{\simeq} TC(S/G; \mathbb{F}_p).$$

In other words, $K^{\mathrm{inf}}(S/G; \mathbb{F}_p) \simeq 0$.

Proof. For weakly regular, stably coherent rings R, the map $K^{\mathrm{cn}}(S) \to K(S)$ from connective K-theory to K-theory is an equivalence by [AMM22, Proposition 2.4]. Moreover, for strictly henselian rings R of residue characteristic p, the map $K^{\mathrm{cn}}(S; \mathbb{F}_p) \to TC(S; \mathbb{F}_p)$ is an equivalence by [CMM21, Theorem 6.1]. Altogether, we have an equivalence

$$K(S; \mathbb{F}_p) \xrightarrow{\simeq} TC(S; \mathbb{F}_p).$$

The rest of the proof of Lemma 2.5 then works word for word with k replaced by R, using that G has decomposable representation theory by assumption. $\square$

2.2. Rational Goodwillie–Jones trace

Consider the trace map $K(-; \mathbb{Q}) \to HC^-(-/\mathbb{Q})$ from rational K-theory to $\mathbb{Q}$-linear negative cyclic homology. We prove that on $\mathcal{B}_{\mathbb{Q}}$, the rational Goodwillie–Jones trace induces an isomorphism in degree zero in Theorem 2.11. We also deduce a counterintuitive direct sum decomposition for positive degrees in Proposition 2.15.

Setup 2.10. Throughout this subsection we take $k = \mathbb{Q}$. Let G be a nice group over $\mathbb{Q}$. Conditionally, the results generalize to any reductive group G over $\mathbb{Q}$.

2.2.1. Rational Goodwillie–Jones trace in degree zero. We start by discussing the following degree zero isomorphism.

Theorem 2.11 (Rational Goodwillie–Jones trace in degree zero)**.** *For any $X \in \mathcal{B}^G_{\mathbb{Q}}$, the trace induces a ring isomorphism isomorphism*

$$K_0^G(X;\mathbb{Q}) \xrightarrow{\cong} (HC^-)_0^G(X/\mathbb{Q}).$$

Proof. The fiber $K^{\inf}(-;\mathbb{Q}) = \mathrm{fib}(K(-;\mathbb{Q}) \to HC^-(-/\mathbb{Q}))$ is a truncating invariant [LT19, proof of Corollary 3.9]. We check that $K_0^{\inf}(BG;\mathbb{Q}) \cong 0 \cong K_{-1}^{\inf}(BG;\mathbb{Q})$ in Lemma 2.12 below. We then apply Theorem 1.16 to $K^{\inf}(-;\mathbb{Q})$ for $i = 0, -1$ separately. We deduce that for all $X \in \mathcal{B}^G_k$ it holds that $K_0^{\inf}(X/G;\mathbb{Q}) \cong 0 \cong K_{-1}^{\inf}(X/G;\mathbb{Q})$, so we get back the desired claim from the long exact sequence associated to the fiber sequence $K^{\inf}(-;\mathbb{Q}) \to K(-;\mathbb{Q}) \to HC^-(-/\mathbb{Q})$ on X/G. □

Again, we still need to supply the case of an equivariant point.

Lemma 2.12. *We have $K_0^{\inf}(BG;\mathbb{Q}) \simeq 0 \simeq K_{-1}^{\inf}(BG;\mathbb{Q})$.*

Proof. Since $HH(\mathrm{pt}/\mathbb{Q}) \simeq \mathbb{Q}$ sitting in degree zero and $HH(-/\mathbb{Q})$ is finitary, we deduce from (1.1) that $HH(BG/\mathbb{Q}) \simeq R(G;\mathbb{Q})$ is the rationalized representation ring sitting in degree zero. Now by the classical construction of $HC^-(-/\mathbb{Q})$ from $HH(-/\mathbb{Q})$ as in [Lod97, §5.1.7], which is compatible with the categorical definition by [Hoy18], we see that $HC^-_\bullet(BG/\mathbb{Q}) \cong HC^-_0(BG/\mathbb{Q})[u]$ as rings with the free generator u sitting in degree -2. In other words, $HC^-(-/\mathbb{Q})$ commutes with the direct sum decomposition (1.1). The same is true for $K(-;\mathbb{Q})$, as it is finitary.

We display the following piece of the associated long exact sequence on BG:

$$HC_1^-(BG/\mathbb{Q}) \to K_0^{\inf}(BG;\mathbb{Q}) \to K_0(BG;\mathbb{Q}) \to HC_0^-(BG/\mathbb{Q}) \to K_{-1}^{\inf}(BG;\mathbb{Q}) \to K_{-1}(BG;\mathbb{Q})$$

Now, $HC_1^-(BG/\mathbb{Q}) \cong 0$ by above, the trace induces an isomorphism $K_0(BG;\mathbb{Q}) \cong R(G;\mathbb{Q}) \cong HC_0^-(BG/\mathbb{Q})$, and $K_{-1}(BG;\mathbb{Q}) \cong 0$ by smoothness. Therefore $K_0^{\inf}(BG;\mathbb{Q}) \cong 0 \cong K_{-1}^{\inf}(BG;\mathbb{Q})$ as desired. □

Remark 2.13. We also deduce the corresponding claim for the cdh-sheaffified theories from Appendix A: for all $X \in \mathcal{B}^G_k$, we have an equivalence

$$KH_0^G(X;\mathbb{Q}) \xrightarrow{\cong} (L_{\mathrm{cdh}}HC^-)_0^G(X/\mathbb{Q}).$$

This is a weaker claim, where both sides are more computable.

Remark 2.14. Under the geometric interpretation of equivariant negative cyclic homology via derived loop stacks [BFN10; Che20; HSS17; Toë14] using the fixed point scheme notation [Löw24, §1], we can rewrite Theorem 2.11 as

$$K_0^G(X;\mathbb{Q}) \xrightarrow{\cong} \pi_0(\mathrm{R\Gamma}(\mathrm{Fix}^{\mathbf{L}}_{\frac{G}{G}}(X), \mathcal{O})^{hS^1})$$

In words, it gives an isomorphism between the zeroeth equivariant K-group of X and homotopy-S^1-invariant global functions on the associated derived fixed-point scheme.

2.2.2. A decomposition for positive degrees. Let us complement Theorem 2.11 with a slightly counterintuitive result on the positive homotopical degrees. In general, algebraic K-theory can be described by gluing the homotopy-invariant contribution of $KH(-)$ with the contribution of $HC^-(-/\mathbb{Q})$ along their maps to the cdh-sheafification $L_{\mathrm{cdh}}HC^-(-/\mathbb{Q})$, which can be regarded as a truncating invariant. We recall this thoroughly in Appendix A.

It turns out that for simple varieties, the gluing condition is vacuous in positive degrees – their positive K-groups decompose into the direct sum of $KH(-)$ and $HC^-(-/\mathbb{Q})$.

Proposition 2.15. *Let $X \in \mathcal{B}^G_{\mathbb{Q}}$, then*

$$K_i^G(X) \cong KH_i^G(X) \oplus (HC^-)_i^G(X/\mathbb{Q}), \qquad \forall i \geqslant 1$$

Proof. From the homotopy fiber square of Construction A.1 defining $L_{\mathrm{cdh}}HC^-(-)$ it suffices to prove

$$(L_{\mathrm{cdh}}HC^-)_i^G(X/\mathbb{Q}) \cong 0, \qquad \forall i \geqslant 1$$

This is the case for $X = \mathrm{pt}$ since $L_{\mathrm{cdh}}HC^-_\bullet(BG/\mathbb{Q}) \cong HC^-_\bullet(BG/\mathbb{Q}) \cong HC^-_0(BG/\mathbb{Q})[u]$ vanishes for all $i \geqslant 1$. We conclude by applying Theorem 1.16 to the $\mathbb{Q}$-linear truncating invariant $L_{\mathrm{cdh}}HC^-(-/\mathbb{Q})$. □

2.3. Equivariant formality for homotopy invariant K-theory

We now move towards applications to homotopy invariant algebraic K-theory KH. We first record a strong equivariant formality statement in Theorem 2.17. We then discuss two easy consequences: the comparison to its topological counterpart K_{top} over $\mathbb{C}$ in §2.3.2 and a very elementary instance of Parshin vanishing over $\mathbb{F}_q$ in §2.3.3.

Setup 2.16. Throughout this subsection, let k be a base field and G a nice group over it. Conditionally, the results generalize to any linearly reductive group G over k.

2.3.1. Equivariant formality for KH. Let us spell out the results from §1.5 for KH. This yields a strong equivariant formality statement for homotopy invariant K-theory of simple varieties.

Theorem 2.17 (KH of simple varieties)**.** *Let k be any base field. For any $X \in \mathcal{B}_k^G$ we have a natural ring isomorphism*

$$KH_0^G(X) \underset{K_0(\mathrm{pt})}{\otimes} K_\bullet(\mathrm{pt}) \xrightarrow{\cong} KH_\bullet^G(X).$$

Furthermore, $KH_0^G(X)$ is a finite projective module over the representation ring $KH_0^G(\mathrm{pt}) \cong R(G)$.

Proof. Follows from Theorem 1.17 as $KH(-)$ is a finitary truncating invariant which is multiplicative. □

Remark 2.18 (Faithfull flatness)**.** Concisely, the above theorem shows that $KH^G(X)$ is connective and finite faithfully flat over $K^G(\mathrm{pt})$.

Remark 2.19 (Change of group and equivariant formality)**.** The equivalence of Theorem 2.17 is compatible with changing the group through any group homomorphism $G' \to G$. In particular, restricting to the trivial group $1 \to G$ specializes to non-equivariant K-theory by base change along the augmentation map

$$R(G) \to \mathbb{Z}\,.$$

This is an *equivariant formality* statement for KH of varieties $X \in \mathcal{B}_k^G$. It allows us to compute non-equivariant KH directly from the equivariant one, where more techniques are available.

Remark 2.20 (Arbitrary base)**.** Let G be a group with decomposable representation theory over an arbitrary base ring R (e.g. over $\mathbb{Z}$). The same arguments show that for any $X_R \in \mathrm{Sch}_R^G$ we have the relative statement

$$KH_0^G(X_R) \underset{KH_0(R)}{\otimes} KH_\bullet(R) \xrightarrow{\cong} KH_\bullet^G(X_R)$$

functorially in base change along ring maps $R' \to R$. Indeed, the non-equivariant base case of $X_R = \operatorname{Spec} R$ is tautological, while the rest of our arguments goes through without change.

For the final remarks, let us focus on the case of a split torus $T = \mathbb{G}_m^n$ over k.

Remark 2.21 (Freeness)**.** For any $X \in \mathcal{B}_k^T$, we deduce that the rationalization $KH_0^T(X;\mathbb{Q})$ is a finite free module over $\mathbb{Q}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.

Indeed, it is a finite projective module over $KH_0^T(\mathrm{pt};\mathbb{Q}) \cong \mathbb{Q}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ by Corollary 2.17. Any finite projective module over $\mathbb{Q}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is free by [Swa78] or [Lam06, §V.4, Corollaries V.4.10 and V.4.11].

Remark 2.22 (Equivariant localization)**.** We expect Theorem 2.17 to be useful in conjunction with equivariant localization from [KR24, Theorem C, §11] for explicit computations of the rings $KH_\bullet^T(X)$ where $X \in \mathrm{Sch}_k^T$.

Indeed, given a T-equivariant variety X, equivariant localization postulates that the natural restriction map between $KH^T(X)$ and $KH^T(X^T)$ becomes an equivalence after localization on the base $K_0^T(\mathrm{pt})$; the latter object is often easy to describe in terms of the T-equivariant geometry. Once equivariant formality holds, $KH^T(X)$ in particular embeds into its localization. The computational question then becomes to describe its image.

For cohomology, the above recipe is well-known and useful. For KH of singular varieties, not much has been done. For example, to the best of our knowledge, there are basically no results on equivariant formality of KH for singular varieties in the literature to start with.

2.3.2. Isomorphisms with topological theories in degree zero. If we work over the complex numbers $k = \mathbb{C}$, we can compare to topological K-theory. More precisely, we have maps of localizing invariants

$$K(-) \to KH(-) \to K_{\mathrm{st}}(-) \to K_{\mathrm{top}}(-)$$

of $\mathbb{C}$-linear finitary localizing invariants [Bla16]. Moreover, $KH(-)$, $K_{\mathrm{st}}(-)$, $K_{\mathrm{top}}(-)$ are truncating.

These three latter theories are believed to be reasonably close to each other, but again examples do not come in plenty. They are indeed close for simple varieties: their degree zero parts[3] are actually isomorphic (while the rest is determined by their value on the point).

Proposition 2.23 (KH and K_{top} of complex varieties)**.** *Let $X \in \mathcal{B}^G_{\mathbb{C}}$, then the natural maps induce ring isomorphisms*

$$KH^G_0(X) \to (K_{\mathrm{st}})^G_0(X) \to (K_{\mathrm{top}})^G_0(X).$$

Proof. This is clearly the case for the non-equivariant point $\mathrm{pt} = \operatorname{Spec}\mathbb{C}$, where all three terms take the value $\mathbb{Z}$. We conclude by Theorem 1.16 and Lemma 1.11. □

Remark 2.24 (Equivariant formality)**.** Note that Theorem 1.17 applies equally well to the truncating finitary multiplicative invariants $K_{\mathrm{st}}(-)$, $K_{\mathrm{top}}(-)$ as to $KH(-)$ in §2.3. All of them are equivariantly formal for any $X \in \mathcal{B}^G_{\mathbb{C}}$, compatibly with the comparison maps. Similarly for the non-finitary truncating invariant $HP(-/\mathbb{C})$.

Remark 2.25 (Topological and arithmetic parts of KH)**.** Paired with the equivariant formality from previous remark, Proposition 2.23 allows to compute $KH^G_\bullet(X)$ by tensoring two independent contributions: the topological contribution of $K^G_{\mathrm{top},0}(X)$ and the arithmetic contribution of $K_\bullet(\mathbb{C})$.

To free the careful reader of any potential doubts, we would like to emphasize that $K^G_{\mathrm{top}}(-)$ is the classical invariant one hopes for.

Remark 2.26 (Blanc vs. Segal)**.** The equivariant topological K-theory K^G_{top} agrees with the classical construction [Seg68] on any $X \in \mathrm{Sch}^G_{\mathbb{C}}$.

Denote $M \subseteq G(\mathbb{C})$ the maximal compact subgroup of the topological group of complex points of G with the analytic topology. Then M is a compact real Lie group and it acts on the topological space $X(\mathbb{C})$ of complex points of X with the analytic topology. In this situation, [Seg68] considered the equivariant topological K-theory spectrum

$$K^M_{top}(X(\mathbb{C})).$$

For X in $\mathrm{Sch}^G_{\mathbb{C}}$, we have a natural comparison map

$$K^G_{\mathrm{top}}(X) \to K^M_{\mathrm{top}}(X(\mathbb{C})).$$

constructed in [HP20, §2.1.3].

Now, both sides have proper excision: for K^G_{top} this holds because it is a truncating invariant via [ES21; LS25], while for K^M_{top} this follows from the long exact sequence of a pair. By equivariant resolution of singularities in characteristic zero (see [EKS25, §6.1] for a discussion), it is thus enough to check that the comparison map is an equivalence on smooth G-equivariant schemes. This has been done in [HP20], [Bla16].

For the sake of completeness, let us mention the following fact: the topological realization of the trace map is always an isomorphism.

Remark 2.27 (Topological Chern character)**.** On topological K-theory, the trace map induces the topological Chern character

$$K_{\mathrm{top}}(-) \otimes \mathbb{C} \to HP(-/\mathbb{C}).$$

For any $X \in \mathrm{Sch}^G_{\mathbb{C}}$ we have an equivalence

$$K^G_{\mathrm{top}}(X) \otimes \mathbb{C} \xrightarrow{\simeq} HP^G(X/\mathbb{C}).$$

For smooth schemes, this is due to [Kon21]. It was extended to smooth quotient stacks by [HP20]. Also see [ES21, Theorem 5.3.2]. For possibly singular stacks with nice stabilizers, see [Kha23]. This also works for linearly reductive G conditionally on [LS25].

[3] To avoid any potential confusion, we emphasize that in our conventions, $K_{\mathrm{top}}(X)$ matches the topological K-*cohomology* built up from vector bundles. In particular, $K_{\mathrm{top},0}(X)$ matches the zeroeth topological K-cohomology group (which is usually denoted with a superscript).

2.3.3. Examples of the equivariant singular Parshin property. We also mention the following silly consequence of the formality of KH from Theorem 2.17. If $k = \mathbb{F}_q$ is a finite field of characteristic p with q elements, each $X \in \mathcal{B}_{\mathbb{F}_q}$ satisfies Parshin's property:

$$K_i(X; \mathbb{Q}) \cong 0, \qquad \forall i \geqslant 1.$$

This property is usually conjectured for smooth projective varieties over $\mathbb{F}_q$ and such claims are open in general. It is certainly expected for singular projective varieties over $\mathbb{F}_q$ – the statement is compatible with cdh descent, so the singular case would follow from the smooth case if one had resolution of singularities for projective varieties over $\mathbb{F}_q$.

In fact, this argument works even equivariantly.

Observation 2.28. *Let G be a nice group over $\mathbb{F}_q$ and (X, Y, Z, E) be an abstract blowup square in $\mathrm{Sch}^G_{\mathbb{F}_q}$. If the Parshin's property*

$$K_i^G(-; \mathbb{Q}) \cong 0, \qquad \forall i \geqslant 0$$

holds for Y, Z, E, then it holds for X as well.

Proof. Since $K^G(X; \mathbb{Q}) \simeq KH^G(X; \mathbb{Q})$ by [KR18, Theorem 1.3.(2)] and [Hoy20, Theorem 1.3.(4)], it satisfies cdh descent, so we conclude from the associated long exact sequence. ☐

However, not many instances of this method are available in practice. The equivariant formality of KH from Theorem 2.2 now easily implies that varieties from $\mathcal{B}^G_k$ qualify.

Example 2.29. Let G be a nice group over $\mathbb{F}_p$. For any $X \in \mathcal{B}^G_{\mathbb{F}_q}$, we have

$$K_i^G(X; \mathbb{Q}) \cong 0, \qquad \forall i \neq 0.$$

In particular, X satisfies Parshin's property.

Proof. To this end, compute

$$K^T_\bullet(X; \mathbb{Q}) \cong KH^T_\bullet(X; \mathbb{Q}) \cong KH^T_0(X) \underset{K_0(\mathrm{pt})}{\otimes} K_\bullet(\mathrm{pt}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \cong KH^T_0(X; \mathbb{Q}).$$

Here, the first isomorphism holds by [KR18, Theorem 1.3.(2)] and [Hoy20, Theorem 1.3.(4)] already with $\mathbb{Z}[\frac{1}{p}]$-coefficients; in the non-equivariant case this goes back to [Wei89; TT90]. The second isomorphism comes from Corollary 2.17 and the final isomorphism from the knowledge that $K_\bullet(\mathrm{pt}; \mathbb{Q})$ is concentrated in degree zero with value $\mathbb{Q}$ by Quillen's work [Wei13, Corollary IV.1.13]. ☐

On the other hand, we can illustrate Observation 2.28 on the following example, which is not directly covered by the class $\mathcal{B}^G_{\mathbb{F}_q}$.

Example 2.30. Split toric varieties over $\mathbb{F}_p$ satisfy the equivariant Parshin's property.

Proof. Note that this is the case for the point; it then holds for smooth split toric varieties by [VV03, Theorem 6.2]. Since the statement is compatible with enlarging T along precomposition with a group homomorphism by Remark 1.6, the case of singular split toric varieties reduces to the smooth case via toric resolutions and cdh descent. ☐

3. Examples of simple varieties

We now pay our debt to the reader by providing examples of varieties for which the above theory applies. Although such varieties are rather special, we recover examples of interest in geometric representation theory. After some basic examples in §3.1, we discuss both finite and affine Schubert varieties §3.2, 3.3 for GL_n. The singularities of such varieties are of deep interest in geometric representation theory.

3.1. Some basic examples

We start with a few elementary examples. Let k be a base field and G a linearly reductive group over it.

Example 3.1 (Partial flag varieties)**.** Projective spaces $\mathbb{P}^n$, Grassmannians $\mathrm{Gr}(n,d)$ and all partial flag varieties $\mathrm{Flag}(n,d_\bullet)$ for GL_n lie in $\mathcal{B}_k$ and hence in $\mathcal{C}_k$ by (1) and (2). Picking any G-action on the underlying vector space V, these G-equivariant varieties lie in $\mathcal{B}_k^G$ and hence in $\mathcal{C}_k^G$.

Example 3.2 (Hirzerbruch surfaces)**.** Let $m,d \geqslant 1$. Consider a direct sum of line bundles $\mathcal{E} = \mathcal{O}(i_1) \oplus \ldots \mathcal{O}(i_m)$ on $\mathbb{P}_k^d$ and let $X := \mathbb{P}_{\mathbb{P}^d}(\mathcal{E})$ be its projectivization. All of this further carries an obvious action of the torus $T = \mathbb{G}_m^{d+1}$. Then $X \in \mathcal{B}_k^T$ by (1) and (2).

It is known that this construction yields all smooth projective toric varieties of Picard rank 2. For example, specializing to $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-n)$ on $\mathbb{P}^1$ yields Hirzerbruch surfaces Σ_n.

Example 3.3 (Cusp)**.** Let X be the projective cuspidal curve over k. Then $X \in \mathcal{B}_k$. Indeed, this follows from the abstract blowup square

$$\begin{tikzcd} \mathbb{P}^1 \arrow[d] & \mathrm{pt} \arrow[l, hook'] \arrow[d] \\ X & \mathrm{pt} \arrow[l, hook'] \end{tikzcd}$$

whose other terms lie in $\mathcal{B}_k$ by (1) and (2). We conclude via (4) because the upper horizontal map is split by the structure map $\mathbb{P}^1 \to \mathrm{pt}$.

Example 3.4 (Node)**.** Let X be the projective nodal curve over k. Then $X \in \mathcal{C}_k$, but not in $\mathcal{B}_k$. Indeed, there is an abstract blowup square

$$\begin{tikzcd} \mathbb{P}^1 \arrow[d] & \mathrm{pt} \sqcup \mathrm{pt} \arrow[l, hook'] \arrow[d] \\ X & \mathrm{pt} \arrow[l, hook'] \end{tikzcd}$$

whose other terms lie in $\mathcal{C}_k$ by (1) and (2). Hence also X lies in $\mathcal{C}_k$ by Definition 1.4.

On the other hand, if X was in $\mathcal{B}_k$, then we would also have $KH_{-1}(X) \cong 0$ by Theorem 2.17. However, the long exact sequence associated with the abstract blowup square above ends with

$$\begin{tikzcd} {} \arrow[r] & KH_0(\mathbb{P}^1) \oplus KH_0(\mathrm{pt}) \arrow[r, "\mathrm{rank}"] \arrow[d, equal] & KH_0(\mathrm{pt} \sqcup \mathrm{pt}) \arrow[r] \arrow[d, equal] & KH_{-1}(C) \arrow[r] \arrow[d, equal] & 0 \\ {} \arrow[r] & \mathbb{Z}^{\oplus 3} \arrow[r, "{(a,b,c) \mapsto (a+b+c, a+b+c)}"] & \mathbb{Z}^{\oplus 2} \arrow[r] & KH_{-1}(C) \arrow[r] & 0 \end{tikzcd}$$

showing $KH_{-1}(X) \cong \mathbb{Z}$. Altogether, $X \notin \mathcal{B}_k$.

Example 3.5 (Closed gluing)**.** If X is covered by two closed subvarieties Z_1 and Z_2 with (reduced) intersection $Z_3 := Z_1 \cap Z_2$ such that $Z_1, Z_2, Z_3 \in \mathcal{C}_k$, then also $X \in \mathcal{C}_k$. Similarly for the equivariant case. Indeed, this immediately follows from the associated abstract blowup square featuring $(X, Z_1 \sqcup Z_2, Z_3, Z_3 \sqcup Z_3)$ and from Definition 1.4.

Example 3.6 (Closure on projective cones)**.** Let k be a field of characteristic zero and E be a smooth projective variety over k. Assume that $E \in \mathcal{B}_k$ (resp. $\mathcal{C}_k$). Let $i : E \hookrightarrow \mathbb{P}_k$ be any projective embedding, $\mathcal{L} = i^* \mathcal{O}(1)$ the corresponding line bundle on E, and X the associated projective cone. Then $X \in \mathcal{B}_k$ (resp. $\mathcal{C}_k$) as well. Similarly for the equivariant case.

Indeed, blowing up X in the cone point, we obtain an abstract blowup

$$\begin{tikzcd} Y \arrow[d] & E \arrow[l, hook', "0"'] \arrow[d] \\ X & \mathrm{pt} \arrow[l, hook'] \end{tikzcd}$$

where $Y = \mathbb{P}_E(\mathcal{O} \oplus \mathcal{L})$ and the upper horizontal map is the inclusion of the zero section. See [CHWW13, Lemma 2.2] – strictly speaking, the reference discusses affine cones, but the case of projective cones follows. We then have $E \in \mathcal{B}_k$ by assumption, hence $\mathbb{P}_E(\mathcal{O} \oplus \mathcal{L}) \in \mathcal{B}_k$ by (2). Also $\mathrm{pt} \in \mathcal{C}_k$ by (1). Since the structure map $\mathbb{P}_E(\mathcal{O} \oplus \mathcal{L}) \to E$ provides a retraction of the upper horizontal embedding, we deduce that $X \in \mathcal{B}_k$ by (4).

Example 3.7 (Projective cone of $\mathbb{P}^1$)**.** Let $k = \mathbb{Q}$. To illustrate that the decomposition from Proposition 2.15 can be nontrivial, we recall the example of the projective cone X of $\mathbb{P}^1$ under the embedding by $\mathcal{O}(2)$, heavily based on the results of [CHWW13] for the affine cone. Note that X accidentally matches the affine Schubert variety $X_{\leqslant 2\omega_1} \hookrightarrow \mathbf{Gr}_{GL_2}$ by [Zhu17, Lemma 2.1.14]. Also compare to [Pan03].
We claim that

$$K_i(X) = \begin{cases} KH_i(X) \oplus HC_i^-(X) & \text{if } i \geqslant 1 \\ KH_0(X) & \text{if } i = 0 \\ 0 & \text{if } i \leqslant -1 \end{cases}$$

where

$$KH_i(X) \cong KH_i(\mathbb{Q})^{\oplus 3} \quad \text{if } i \geqslant 0 \qquad \text{and} \qquad HC_i^-(X) \cong \begin{cases} 0 & \text{if } i \geqslant 1 \text{ even,} \\ \mathbb{Q} & \text{if } i \geqslant 1 \text{ odd.} \end{cases}$$

To get this, write $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(2))$ for the resolution of X. This matches the affine Demazure resolution $Y_{\leqslant(\omega_1,\omega_1)}$, but we do not need this fact. We stratify X into the affine cone X_1 and the complement of the singular point X_2, and write $X_3 = X_1 \cap X_2$ for their intersection. Pulling back this stratification to Y, we get the diagram

$$\begin{tikzcd} Y \arrow[d] & Y_1 \sqcup Y_2 \arrow[l] \arrow[d] & Y_3 \arrow[l, shift left] \arrow[l, shift right] \arrow[d] \\ X & X_1 \sqcup X_2 \arrow[l] & X_3 \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

where rows are Zariski covers. The maps $Y_2 \to X_2$ and $Y_3 \to X_3$ are isomorphisms. We now apply $K(-)$ to this diagram; rows will then induce long exact sequences by Zariski descent.

$$\begin{tikzcd} K(Y) \arrow[r] & K(Y_1) \oplus K(Y_2) \arrow[r] & K(Y_3) \\ K(X) \arrow[r] \arrow[u, "f^*"] & K(X_1) \oplus K(X_2) \arrow[r] \arrow[u, "f_1^* \oplus \mathrm{id}"] & K(X_3) \arrow[u, "\mathrm{id}"] \end{tikzcd}$$

The upper row consists of smooth varieties, and we have $K(Y) = K(\mathbb{Q})^{\oplus 4}$ by repeated use of the projective bundle formula. From the split abstract blowup square associated to the resolution $f : Y \to X$, we deduce $KH(X) = K(\mathbb{Q})^{\oplus 3}$ is a direct summand in $KH(Y) = K(Y)$. Since X_2 and X_3 are smooth, the remaining difference between $K(X)$ and $K(Y)$ comes from $K(X_1)$, which is described by [CHWW13, Theorem 4.3 and Remark 4.3.1] – it gives $\mathbb{Q}$ in odd positive degrees $2j + 1 \geqslant 1$ and zero else. However, each of these $\mathbb{Q}$ maps to 0 under $f_1^* : K_{2j+1}(X_1) \to K_{2j+1}(Y_1)$ – indeed, $\mathbb{Q}$ is uniquely divisible, while $K_{2j+1}(Y_1) \cong K_{2j+1}(\mathbb{Q})^{\oplus 2}$ is torsion for $2j + 1 \geqslant 3$ by [Wei13] and $(\mathbb{Q}^\times)^{\oplus 2}$ for $2j + 1 = 1$ (alternatively one can keep track of the weights). From the commutativity the right-hand square and the fact that the right vertical map is the identity, we now see that the whole $K_{2j+1}(X_1) \cong \mathbb{Q}$ maps to zero under $K_{2j+1}(X_1) \to K_{2j+1}(X_3)$ – hence it contributes to $K_{2j+1}(X)$ by long exact sequence induced by the bottom row. We have thus matched $K_i(X) = KH_i(X) = KH(\mathbb{Q})^{\oplus 3}$ when i is not odd positive; there is an extra $\mathbb{Q}$ appearing if i is odd positive.

On the other hand, we already know that $K_i(X) \cong KH_i(X) \oplus HC_i^-(X)$ for all $i \geqslant 1$ by Proposition 2.15. Altogether, we deduce the result.

3.2. Classical Schubert varieties

We prove that finite Schubert varieties in usual Grassmannians are simple T-equivariant varieties in Lemma 3.10. Already here, our computations seem to be new.

We start by quickly recalling the definitions; see [Bri05] or [Oet21, §2.2, §2.3] for these standard facts. For the sake of readability, we stick to the convention that $\mathrm{Grass}^*(-, d)$ parametrizes rank d locally free subbundles throughout this subsection.

Setup 3.8. Let k be a base field and U a k-vector space of dimension n. Let $F_\bullet U = (0 = F_0 U \subseteq F_1 U \subseteq F_2 U \subseteq \cdots \subseteq F_n U = U)$ be a fixed full reference flag in U. We denote by $B \leqslant GL_n$ the Borel subgroup stabilizing $F_\bullet U$ and by $T = \mathbb{G}_m^{\times n}$ its maximal torus. Given $d \geqslant 0$, we denote $\mathrm{Grass}_k^*(U, d)$ the Grassmannian of d-planes in U.

Recollection 3.9 (Finite Schubert varieties)**.** Fix a sequence of non-negative integers $j_\bullet = (0 = j_0 \leqslant j_1 \leqslant \cdots \leqslant j_n = d)$ with $j_i \leqslant i$ for each $i = 0, \ldots, n$. Denote $d_\bullet = (d_1, \ldots, d_n)$ the corresponding difference sequence with $d_i := j_i - j_{i-1}$ for each $i = 1, \ldots, n$. The associated *finite Schubert variety* $X := X_{\leqslant j_\bullet}$ is the reduced closed subvariety of $\mathrm{Grass}_k^*(U, d)$ classifying

$$X_{\leqslant j_\bullet} = \{V \subseteq U \mid \dim_k V = d,\ \forall i:\ \dim_k(V \cap F_i U) \geqslant j_i\}.$$

It is a projective variety with a natural action of $T \leqslant B$. Set-theoretically, it is a closed union of B-orbits in $\mathrm{Grass}_k^*(U, d)$. It is often singular (the singular cases being determined by explicit combinatorial criteria).

Lemma 3.10 (Finite Schubert varieties are simple)**.** *For any d and $j_\bullet$ as above, we have $X \in \mathcal{B}_k^T$.*

Proof. Consider the Bott–Samelson resolution $Y := Y_{\leqslant j_\bullet}$, which classifies

$$Y_{\leqslant j_\bullet} = \{(0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n) \mid V_i \subseteq F_i U,\ \forall i:\ \dim_k V_i = j_i\}$$

The map $f : Y_{\leqslant j_\bullet} \to X_{\leqslant j_\bullet}$ given by $(V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n) \mapsto V_n$ is proper birational and T-equivariant.

On one hand, $Y_{\leqslant j_\bullet}$ is a tower of honest Grassmannian bundles over pt, defined inductively as follows: in the i-th step, we consider the Grassmannian classifying d_i-dimensional subspaces in $F_i U$ modulo the tautological bundle with fiber V_{i-1} from the previous step.

On the other hand, the map $f : Y_{\leqslant j_\bullet} \to X_{\leqslant j_\bullet}$ can be constructed as a tower of stratified Grassmannian bundles, defined inductively as follows. Denote $\mathcal{V}$ the tautological bundle on $X_{\leqslant j_\bullet}$ with fiber V. In the i-th step, we consider the Grassmannian classifing d_i-dimensional subspaces in the pullback of $\mathcal{V} \cap F_\bullet U_i$ modulo the tautological subbundle from the previous step (also note that by definition of $X_{\leqslant j_\bullet}$ and $\mathcal{V}$, this coherent sheaf has rank $\geqslant d_i$ everywhere).

Altogether, we get a T-equivariant diagram

$$X \xleftarrow{f} Y \xrightarrow{g} \mathrm{pt}$$

where f is an iterated stratified Grassmannian bundle with nonempty fibers and g is an honest iterated Grassmannian bundle. Since $\mathrm{pt} \in \mathcal{B}_k^T$ by (1), also $Y \in \mathcal{B}_k^T$ by iterative use of (2). Hence $X \in \mathcal{B}_k^T$ by iterative use of (3), the relevant rank condition being satisfied by the above. □

3.3. Affine Schubert varieties

Let k be any base field. Consider the split group GL_n over k with its diagonal torus T. The motivating examples of our interest are the affine Schubert varieties in the GL_n affine Grassmannian $\mathbf{Gr}$. These are certain equivariant singular projective varieties; they are of considerable interest in geometric representation theory. We show in Lemma 3.14 that they lie in $\mathcal{B}_k^{GL_n}$, and in particular in $\mathcal{B}_k^T$.

Before going to the proof, we quickly recall the relevant geometric input. See [Zhu17] for details. Also [BS17] is helpful (although the geometric setup is different, their description of the Demazure morphism is applicable and very clear).

Recollection 3.11 (Affine Schubert varieties)**.** The GL_n affine Grassmannian [Zhu17, §1.1] is the ind-scheme representing the moduli problem

$$\mathbf{Gr} : R \mapsto \{\Lambda \subseteq R(\!(t)\!)^{\oplus n} \mid \Lambda \text{ full } R[\![t]\!]\text{-lattice}\}.$$

Let $\Lambda_0 = R[\![t]\!] \subseteq R(\!(t)\!)^{\oplus n}$ be the standard lattice. Given any dominant coweight $\mu \in X_\bullet^+(T)$, we obtain the corresponding affine Schubert variety $X_{\leqslant\mu}$. By definition [Zhu17, §2.1], this is the reduced subscheme of $\mathbf{Gr}$ on the closed subfunctor

$$R \mapsto \{\Lambda \subseteq R(\!(t)\!)^{\oplus n} \mid \Lambda \text{ full } R[\![t]\!]\text{-lattice, the relative position of } \Lambda \text{ to } \Lambda_0 \text{ is } \leqslant \mu\}.$$

It is a projective variety over k with an action of the positive loop group L^+GL_n. Hence it carries an action of GL_n and in particular of T.

Recollection 3.12 (The tautological bundle)**.** The variety $X_{\leqslant\mu}$ carries a natural vector bundle $\mathcal{E} = \mathcal{E}_{\leqslant\mu}$ defined as follows (see also [Zhu17, §1.1] and [BS17, §7]). We write $\mathrm{mod}^{\mathrm{fl}}(R[\![t]\!])$ for the abelian category of finite length $R[\![t]\!]$-modules. Without loss of generality, assume that we can write $\mu = \mu_1 + \cdots + \mu_\ell$ as a sum of ℓ fundamental dominant coweights $\mu_i \in X_\bullet^+(T)$ and call $\ell = \lg(\mu)$ the length of μ. Under this assumption, any $\Lambda \in X_{\leqslant\mu}(R)$ lies inside Λ_0. (The assumption is indeed harmless – all $\mu \in X_\bullet^+(T)$ are of this form up to adding a negative multiple $-m\omega_n$ of the determinant coweight, i.e. replacing Λ_0 by $t^{-m}\Lambda_0$.) We then define

$$\begin{aligned} \mathcal{E} : X_{\leqslant\mu}(R) &\to \mathrm{mod}^{\mathrm{fl}}(R[\![t]\!]) \\ \Lambda &\mapsto \Lambda_0/\Lambda. \end{aligned}$$

This gives a vector bundle on $X_{\leqslant\mu}$, carrying a nilpotent operator $t : \mathcal{E} \to \mathcal{E}$. It is naturally L^+GL_n-equivariant. Consider now

$$\mathcal{E}_\bullet := [\mathcal{E} \xrightarrow{t} \mathcal{E}] \qquad \text{and} \qquad \mathcal{F} := \mathcal{H}_0(\mathcal{E}_\bullet) = \mathrm{coker}(\mathcal{E} \xrightarrow{t} \mathcal{E}).$$

Then $\mathcal{E}_\bullet$ is a perfect complex of Tor amplitude $[1, 0]$ resolving the coherent sheaf $\mathcal{F}$.

Recollection 3.13 (Affine Demazure resolutions)**.** Given any sequence $\mu_\bullet = (\mu_1, \ldots, \mu_\ell)$ summing up to μ, we get a corresponding (partial) affine Demazure resolution $Y_{\leqslant\mu_\bullet}$, see [Zhu17, (2.1.17)]. This is the underlying reduced scheme of the moduli problem

$$R \mapsto \{(\Lambda_\ell, \ldots, \Lambda_0) \mid \Lambda \subseteq R(\!(t)\!)^{\oplus n} \text{ full } R[\![t]\!]\text{-lattice, the relative position of } \Lambda_i \text{ to } \Lambda_{i-1} \text{ is } \leqslant \mu_i\}.$$

The *convolution map* $f : Y_{\leqslant\mu_\bullet} \to X_{\leqslant\mu}$ given by $(\Lambda_\ell, \ldots, \Lambda_0) \mapsto \Lambda_\ell$ is equivariant, proper, birational.

In particular, write $\mu = \mu_1 + \lambda$ with μ_1 minuscule and $\lg(\lambda) < \lg(\mu)$. Then the convolution map f is given by the structure map of the stratified Grassmannian bundle

$$Y_{\leqslant(\mu_1,\lambda)} = \mathrm{Grass}_{X_{\leqslant\mu}}(\mathcal{F}, \mu_1) \xrightarrow{f} X_{\leqslant\mu}.$$

On the other hand, since μ_1 is minuscule, $Y_{\leqslant\mu_\bullet}$ is an honest Grassmannian bundle $\mathrm{Grass}_{X_{\leqslant\lambda}}(V, \mu_1)$ over $X_{\leqslant\lambda}$ associated to an n-dimensional vector space V with a L^+GL_n-action.

Altogether, we obtain the convolution diagram

$$X_{\leqslant\mu} \xleftarrow{f} Y_{\leqslant\mu_\bullet} \xrightarrow{g} X_{\leqslant\lambda} \tag{3.1}$$

which is L^+GL_n-equivariant, and in particular GL_n-equivariant.

We are now ready to deduce that affine Schubert varieties are simple.

Lemma 3.14 (Affine Schubert varieties are simple)**.** *Let $X_{\leqslant\mu}$ be any affine Schubert variety in the GL_n affine Grassmannian, then*

$$X_{\leqslant\mu} \in \mathcal{B}_k^{GL_n}.$$

In particular, $X_{\leqslant\mu} \in \mathcal{B}_k^T$.

Proof. We will prove that $X_{\leqslant\mu} \in \mathcal{B}_k^{GL_n}$ by induction on the length of μ.

If $\lg(\mu) \leqslant 1$, then $X_{\leqslant\mu}$ is either the point pt or an honest Grassmannian $\mathrm{Grass}_k(V, \mu_i)$ associated to an n-dimensional vector space V with an action of GL_n. We thus have $X_{\leqslant\mu} \in \mathcal{B}_k^{GL_n}$ in this case by the closure properties (1) and (2).

Now assume $\lg(\mu) \geqslant 2$ and write $\mu = \mu_i + \lambda$ with μ_i minuscule and $\lg(\lambda) < \lg(\mu)$. We get the convolution diagram (3.1)

By induction, we already know that $X_{\leqslant\lambda} \in \mathcal{B}_k^{GL_n}$. Since $Y_{\leqslant\mu_\bullet}$ is an honest Grassmannian bundle over it, we deduce $Y_{\leqslant\mu_\bullet} \in \mathcal{B}_k^{GL_n}$ by the closure property (2). But now $Y_{\leqslant\mu}$ is a stratified Grassmannian bundle over $X_{\leqslant\mu}$, so we deduce $X_{\leqslant\mu} \in \mathcal{B}_k^{GL_n}$ by the closure property (3). This completes the inductive step.

In particular, X lies in $\mathcal{B}_k^T$ and $\mathcal{B}_k$ by Remark 1.6 □

Remark 3.15. The same argument goes through with respect to the groups $T \times \mathbb{G}_m \leqslant GL_n \times \mathbb{G}_m$ extended by the loop rotation action of $\mathbb{G}_m$.

A. Cdh-sheafified (negative) cyclic homology as truncating invariant

For this section, let $k = \mathbb{Q}$ or a finite field extension thereof. It is an important consequence of [KST18; LT19; CHSW08] that on schemes, $K(-)$ is given by gluing the homotopy-invariant contribution of $KH(-)$ with $HC^-(-/k)$ along their maps to the cdh-sheaffification $L_{\mathrm{cdh}}HC^-(-/k)$; this works more generally in any characteristic using $TC(-)$. See [EM23, Diagrams (3.9) and (4.1)] for an overview. Consequently, $L_{\mathrm{cdh}}HC^-(-/k)$ extends to a truncating invariant on $\mathrm{Cat}_k^{\mathrm{perf}}$ as in [EM23, Lemma 5.7]; also see [EKS25, §8.1].

This not only yields a clear extension to the equivariant setup, but further allows arguments based on semi-orthogonal decompositions (whose pieces do not a priori need to come from geometry). Such arguments are useful in practice, so we record the details. The same trick works for cyclic homology; we include this as well.

Construction A.1. Define the localizing invariant $L_{\mathrm{cdh}}HC^-(-/k) : \mathrm{Cat}_k^{\mathrm{perf}} \to \mathrm{Sp}$ by the pushout

$$\begin{tikzcd} K(-) \arrow[r] \arrow[d] & HC^-(-/k) \arrow[d] \\ KH(-) \arrow[r] & L_{\mathrm{cdh}}HC^-(-/k). \end{tikzcd}$$

We use the standard conventions on (co)homological shifts. Namely, the shift [1] corresponds to the suspension Σ; it increases homological degree (and decreases cohomological degree) by 1.

Construction A.2. We define $L_{\mathrm{cdh}}HC(-/k) : \mathrm{Cat}_k^{\mathrm{perf}} \to \mathrm{Sp}$ by the $[-2]$-shift of the bottom fiber sequence in

$$\begin{tikzcd} HC^-(-/k) \arrow[r] \arrow[d] & HP(-/k) \arrow[r] \arrow[d] & HC(-/k)[2] \arrow[d] \\ L_{\mathrm{cdh}}HC^-(-/k) \arrow[r] & HP(-/k) \arrow[r] & L_{\mathrm{cdh}}HC(-/k)[2]. \end{tikzcd}$$

Here, the map $L_{\mathrm{cdh}}HC^-(-/k) \to HP(-/k)$ comes from the defining pushout for $L_{\mathrm{cdh}}HC^-(-/k)$ by noting that that the composition $K(-) \to HC^-(-/k) \to HP(-/k)$ factors through $K(-) \to KH(-)$ since $KH(-)$ is the initial $\mathbb{A}^1$-invariant localizing invariant on $\mathrm{Cat}_k^{\mathrm{perf}}$ by [EKS25, Example 3.2.6.(2)] and $HP(-/k)$ is $\mathbb{A}^1$-homotopy invariant on $\mathrm{Cat}_k^{\mathrm{perf}}$ by the discussion [EKS25, Example 3.2.7.(3)].

Remark A.3. It is an interesting question whether one can similarly define $L_{\mathrm{cdh}}HH(-/k)$ as a truncating invariant of small stable k-linear categories.

Observation A.4. *The above defined $L_{\mathrm{cdh}}HC^-(-/k)$ and $L_{\mathrm{cdh}}HC(-/k)$ are truncating invariants on $\mathrm{Cat}_k^{\mathrm{perf}}$.*

Proof. First note that both $KH(-)$ and $HP(-/k)$ are truncating invariants on $\mathrm{Cat}_k^{\mathrm{perf}}$ by [LT19, Proposition 3.14 and Corollary 3.11] and [EM23, proof of Lemma 5.7]. Furthermore, looking at the first pushout diagram, the fiber $K^{\mathrm{inf}}(-) = \mathrm{fib}(K(-) \to HC^-(-/k)) \simeq \mathrm{fib}(KH(-) \to L_{\mathrm{cdh}}HC^-(-/k))$ is a truncating invariant as well. From the fiber sequence $K^{\mathrm{inf}}(-) \to KH(-) \to L_{\mathrm{cdh}}HC^-(-/k)$ we deduce that $L_{\mathrm{cdh}}HC^-(-/k)$ is truncating. Since $HP(-/k)$ is truncating, the shifted cofiber $L_{\mathrm{cdh}}HC(-/k)$ of $L_{\mathrm{cdh}}HC^-(-/k) \to HP(-/k)$ is also truncating. □

Observation A.5. *If $\mathcal{X} = X/G$ is a global quotient of a regular scheme X by a nice group G, the natural maps induce equivalences*

$$HC^-(\mathcal{X}/k) \xrightarrow{\simeq} L_{\mathrm{cdh}} HC^-(\mathcal{X}/k),$$
$$HC(\mathcal{X}/k) \xrightarrow{\simeq} L_{\mathrm{cdh}} HC(\mathcal{X}/k).$$

Proof. If $\mathcal{X} = X/G$ is a global quotient of a regular scheme by a nice group, the map $K(\mathcal{X}) \to KH(\mathcal{X})$ is an equivalence by [Hoy20, Theorem 1.3, (1) and (4)]; the result then propagates through the defining pushouts and fiber sequences. □

On schemes, the above defined invariants $L_{\mathrm{cdh}} HC^-(-/k)$ and $L_{\mathrm{cdh}} HC(-/k)$ recover the usual cdh-sheaffifications of $HC^-(-/k)$ and $HC(-/k)$. In fact, this works equivariantly with respect to nice groups.

Observation A.6. *Let G be a nice group. When restricted to invariants of Sch_k^G, the above defined truncating invariants $(L_{\mathrm{cdh}} HC^-)^G(-/k)$ and $L_{\mathrm{cdh}} HC^G(-/k)$ are given by the actual cdh sheafifications of $(HC^-)^G(-/k)$ and $HC^G(-/k)$.*

Proof. Since the newly defined invariants $L_{\mathrm{cdh}} HC^-(-/k)$, $L_{\mathrm{cdh}} HC(-/k)$ are truncating by Observation A.4, they in particular satisfy cdh descent on Sch_k^G by [LT19; ES21], hence they get natural maps from the actual cdh-sheaffifications of $HC^-(-/k)$ and $HC(-/k)$ on Sch_k^G. These maps are equivalences on regular schemes by Observation A.5. Since we are working over a field of characteristic zero (a finite field extension of $\mathbb{Q}$), we conclude via cdh descent from the existence of G-equivariant resolutions of singularities – see [EKS25, §6.1] for a brief overview; this argument already appears in [Hoy20, Corollary 1.4]. □

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